NOTE TO FILE
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Three Shapes of AM Revisited

1 References

A. 140927 XLS Atwood's Machine -Three Shapes R3.xls
B. 150409 Family of Maps R1.xlsx
C. 150411 Boyle - Annex 2 - AM Example R2.docx
D.

2 Purpose

To explore the variations in the hump-backed curves coming out of the AM and the RM examples.

3 Background

In September, at Ref A, I attempted to produce the curve of Pinkerton and Odum. I got three curves, the third of which was what I was looking for. I discounted the other two as anomalous, but always had the question in my mind as to why they arose.

While writing Ref C I realized that there was a hidden constraint in the development of the Pinkerton curve, and if I exercised that constraint in the three ways possible, I got three curves.

I discovered, to my surprise, that there are actually three different maximum power curves coming out of an analysis of Atwood's Machine (the AM). Each depends on an arbitrary assumption about which of the three symbols for mass are held constant in the analysis. This assumption amounts to an additional arbitrary constraint on the design changes in the machine as it is modified to test/demonstrate different efficiencies. You can change ML, you can change MH, or you can change both. If you change both randomly, that is not of much interest. In that case, hold MT constant.

I then went back, in this note, to examine the AM and the Resistance Circuits of Jacobi's law to see what I could learn about this phenomenon. At Ref B I examine a set of 750 different unit maps, which form a family of unit maps, all of which have maximum power at an intermediate value of efficiency. However, of the six unit maps arising from the AM and RM examples, only three of the six have this feature.

4 Discussion

4.1 Atwood's Machine

4.1.1 M_T Constant

Average power of the useful energy in the form constant times factor in eta [$P_U(\eta) = C * f(\eta)$]. Here M_T is constant, while M_H and M_L are allowed to vary.

$$\mathrm{P}_U(\eta) = \sqrt[2]{\frac{Dg^3{M_T}^2}{2}} \times \sqrt[2]{\frac{\eta^2(1-\eta)}{(1+\eta)^3}}$$	Equ 4.1.1.1

Reduction to a Unit map:
The right-hand factor is strictly concave on the interval [0, 1]. It has a maximum value of P_{max} = $1/\sqrt{27}$ when $\eta = ½$. So, the vertex is at $V = \left(\frac{1}{2}, {}^{1}\!/\!_{\sqrt[2]{3^3}}\right) = \left(\frac{1}{2}, \sqrt[2]{\frac{1}{27}}\right)$

Define a new function $\phi_U(\eta)$ as follows:

$$\phi_U(\eta) = {}^{\mathrm{P}_U}\!/\!_{P_{max}} = \frac{\sqrt[2]{\frac{Dg^3{M_T}^2}{2}} \times \sqrt[2]{\frac{\eta^2(1-\eta)}{(1+\eta)^3}}}{\sqrt[2]{\frac{Dg^3{M_T}^2}{2}} \times \sqrt[2]{\frac{1}{27}}}$$	Equ 4.1.1.2

After cancelling:

$$\phi_U(\eta) = {}^{\mathrm{P}_U}\!/\!_{P_{max}} = \sqrt[2]{27} \times \sqrt[2]{\frac{\eta^2(1-\eta)}{(1+\eta)^3}}$$	Equ 4.1.1.3

4.1.2 M_H Constant

We have two equations relating the masses.

- MT = MH + ML
- η = ML / MH

Using these I can express any one symbol in terms of any other two symbols. So, if η is variable and one of the mass symbols is constant C, I can express MT as MT = C * f(η). Substituting that into equation 1.1 gives me another variant.

In this case I get: $M_T = M_H(1+\eta)$

Which leads to a new expression for $P_U(\eta)$. Average power of the useful energy in the form constant times factor in eta [$P_U(\eta) = C * f(\eta)$]. Here M_H is constant, while M_T and M_L are allowed to vary.

$P_U(\eta) = \sqrt[2]{\frac{Dg^3{M_H}^2}{2}} \times \sqrt[2]{\frac{\eta^2(1-\eta)}{(1+\eta)}}$	Equ 4.1.2.1

Reduction to a Unit map:

The right-hand factor is strictly concave on the interval [0, 1]. It has a maximum value of $P_{max} = \sqrt[2]{\frac{7-3\sqrt{5}}{1+\sqrt{5}}}$ when $\eta = \frac{\sqrt{5}-1}{2}$. So, the vertex is at $V = \left(\frac{\sqrt{5}-1}{2}, \sqrt[2]{\frac{7-3\sqrt{5}}{1+\sqrt{5}}}\right)$. Just incidentally, this value of η computes to 0.618033988749895. The number γ = 1.618033988749895 is called the golden mean, or the golden ratio, as used in ancient Greek architecture. So the vertex is at the point where η = γ - 1. Now that is interesting, in its own right.

Define a new function $\phi_U(\eta)$ as follows:

$\phi_U(\eta) = {}^{P_U}/_{P_{max}} = \frac{\sqrt[2]{\frac{Dg^3{M_H}^2}{2}} \times \sqrt[2]{\frac{\eta^2(1-\eta)}{(1+\eta)}}}{\sqrt[2]{\frac{Dg^3{M_H}^2}{2}} \times \sqrt[2]{\frac{7-3\sqrt{5}}{1+\sqrt{5}}}}$	Equ 4.1.2.2

$\phi_U(\eta) = {}^{P_U}/_{P_{max}} = \sqrt[2]{\frac{1+\sqrt{5}}{7-3\sqrt{5}}} \times \sqrt[2]{\frac{\eta^2(1-\eta)}{(1+\eta)}}$	Equ 4.1.2.3

4.1.3 M_L Constant

Here I have the substitution: $M_T = M_L * (1 + \eta) / \eta$

Average power of the useful energy in the form constant times factor in eta [$P_U(\eta) = C * f(\eta)$].
Here M_L is constant, while M_T and M_H are allowed to vary.

$$P_U(\eta) = \sqrt[2]{\frac{Dg^3{M_L}^2}{2}} \times \sqrt[2]{\frac{(1-\eta)}{(1+\eta)}}$$	Equ 4.1.3.1

Reduction to a Unit map:
The right-hand factor is not strictly concave on the interval [0, 1]. It has a maximum value of $P_{max} = 1$ when $\eta = 0$. It goes through the points (0, 1) and (1, 0).

Define a new function $\phi_U(\eta)$ as follows:

$$\phi_U(\eta) = {}^{P_U}/_{P_{max}} = \frac{\sqrt[2]{\frac{Dg^3{M_L}^2}{2}} \times \sqrt[2]{\frac{(1-\eta)}{(1+\eta)}}}{\sqrt[2]{\frac{Dg^3{M_L}^2}{2}} \times 1}$$	Equ 4.1.3.2

$$\phi_U(\eta) = {}^{P_U}/_{P_{max}} = \sqrt[2]{\frac{(1-\eta)}{(1+\eta)}}$$	Equ 4.1.3.3

4.2 Resistance Matching (Electronic) Example

4.2.1 R_{int} Constant

In the resistance matching example, we have:

- $R_{total} = R_{int} + R_{load}$
- $\eta = R_{load} / R_{total} = R_{load} / (R_{int} + R_{load})$

Average power of the useful energy in the form constant times factor in eta [$P_U(\eta) = C * f(\eta)$].
Here R_{int} is constant, while R_{total} and R_{load} are allowed to vary.

$$P_U = \frac{{V_S}^2}{R_{int}} \times \eta(1-\eta)$$	Equ 4.2.1.1

Reduction to a Unit map:
The right-hand factor is strictly concave on the interval [0, 1]. It has a maximum value of P_{max} = ¼ when η = ½. So, the vertex is at $V = \left(\frac{1}{2}, \frac{1}{4}\right)$..

Define a new function $\phi_U(\eta)$ as follows:

$$\phi_U(\eta) = {}^{P_U}/_{P_{max}} = \frac{\frac{{V_S}^2}{R_{int}} \times \eta(1-\eta)}{\frac{{V_S}^2}{R_{int}} \times \frac{1}{4}}$$	Equ 4.2.1.2

$$\phi_U(\eta) = {}^{P_U}/_{P_{max}} = 4\eta(1-\eta)$$	Equ 4.2.1.3

This result is generally known as the “Maximum Power Transfer Theorem” or as Jacobi’s Law of electrical circuits.

4.2.2 R_{load} Constant

But, if we are arbitrarily going to hold one of the three values for resistance constant, we can also let R_{load} be constant.

Solving the equations for the three resistances and eta I get:

- $\eta = R_{load} / (R_{int} + R_{load})$
- $\eta (R_{int} + R_{load}) = R_{load}$
- $R_{load} (1 - \eta) = \eta R_{int}$
- $R_{load} = R_{int} ((\eta / (1 - \eta))$
- $R_{int} = R_{load} (1 - \eta) / \eta$

Subbing this into 4.2.1.1 I get:

$$P_U = \frac{{V_S}^2}{R_{load}} \times \eta^2$$	Equ 4.2.2.1

Reduction to a Unit map:
The right-hand factor is convex on the interval [0, 1], being a parabola facing upwards. It has a maximum value of $P_{max} = 1$ when $\eta = 1$.

Define a new function $\phi_U(\eta)$ as follows:

$$\phi_U(\eta) = {}^{P_U}/_{P_{max}} = \frac{\frac{{V_S}^2}{R_{load}} \times \eta^2}{\frac{{V_S}^2}{R_{load}} \times 1}$$	Equ 4.2.2.2

$$\phi_U(\eta) = {}^{P_U}/_{P_{max}} = \eta^2$$	Equ 4.2.2.3

4.2.3 R_{total} Constant

Again, solving the equations for the three resistances and eta I get:

- $R_{int} = R_{total} (1 - \eta)$

Subbing this in I get:

Subbing this into 4.2.1.1 I get:

$$P_U = \frac{{V_S}^2}{R_{total}} \times \eta$$	Equ 4.2.3.1

Reduction to a Unit map:
The right-hand factor is linear on the interval [0, 1], being a simple line. It has a maximum value of $P_{max} = 1$ when $\eta = 1$.

Define a new function $\phi_U(\eta)$ as follows:

$$\phi_U(\eta) = {}^{P_U}/_{P_{max}} = \frac{\frac{{V_S}^2}{R_{total}} \times \eta}{\frac{{V_S}^2}{R_{total}} \times 1}$$	Equ 4.2.3.2

$$\phi_U(\eta) = {}^{P_U}/_{P_{max}} = \eta$$	Equ 4.2.3.3

4.3 Family of Curves??

I note that, by dividing by the P_{max} I am usually converting a non-surjective map into a surjective unit map. But, if I just focus on the variable factor (and ignore the physical scaling constant) it seems clear that there is a family of curves hiding in the background here. So, suppose for this discussion that the physical scaling constant C is = 1, and can be ignored. What we are left with is the right-hand factor, a function of η.

$P_U = C \times \left(\frac{\eta^A (1-\eta)^B}{(1+\eta)^C}\right)^{1/D}$	Equ 4.1

This family of curves are all unit maps, but the members are not necessarily surjective. I.e. they all have a range within the unit interval. The argument goes like this. The numerator is composed of two factors, each of which is less than or equal to 1, so the numerator is less than or equal to 1. The denominator is greater than or equal to 1. So the ratio within the brackets must be less than or equal to 1. The Dth root of a number in the unit interval is also in the unit interval. Therefore, this represents a unit map, which may be surjective or non-surjective.

In order to get a strictly concave zero-bounded map, there are constraints on the symbols:

- $\eta \in \Re$, $\eta \in [0, 1]$
- $A \in N$, $A > 0$
- $B \in N$, $B > 0$
- $C \in N$, $C >= 0$
- $D \in N$, $D > 0$

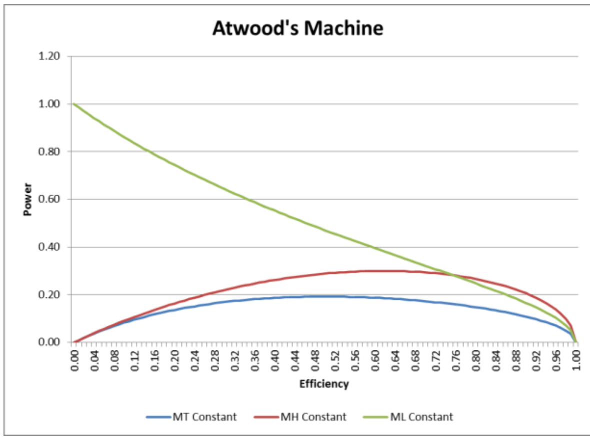
C can be zero, making the denominator = 1, and it disappears. A and B must be greater than zero to make the function pass through (0,0) and (1,0). D cannot be 0 as that would cause division by zero. All are positive. η is real, and the others are natural numbers.

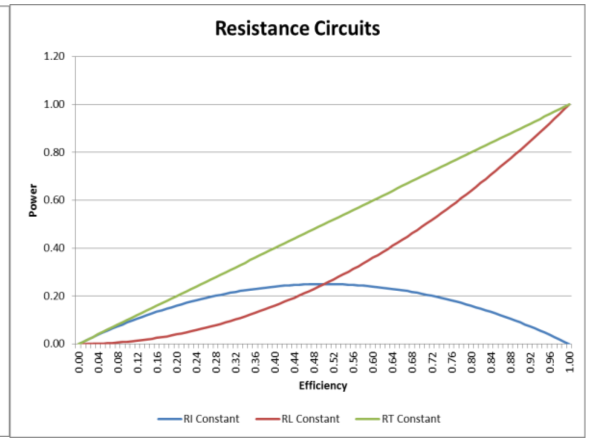
I can designate a member of this family, whether concave zero-bounded or not, as an ordered 4-tuple (A, B, C, D) as follows:

- Atwood's machine – M_T constant – (A, B, C, D) = (2, 1, 3, 2) [Pinkerton 50% curve]
- Atwood's machine – M_H constant – (A, B, C, D) = (2, 1, 1, 2) [Other 60% curve]
- Atwood's machine – M_L constant – (A, B, C, D) = (0, 1, 1, 2) [Not a concave unit map.]
- Resistance Matching – R_{int} constant – (A, B, C, D) = (1, 1, 0, 1) [Jacobi's law]
- Resistance Matching – R_{load} constant – (A, B, C, D) = (2, 0, 0, 1) [Not a concave unit map.]
- Resistance Matching – R_{total} constant – (A, B, C, D) = (1, 0, 0, 1) [Not a concave unit map.]

So, in three out of six instances, I get a concave-downwards unit map at the heart of the dynamics.

The six curves are displayed here in these two JPG graphics.

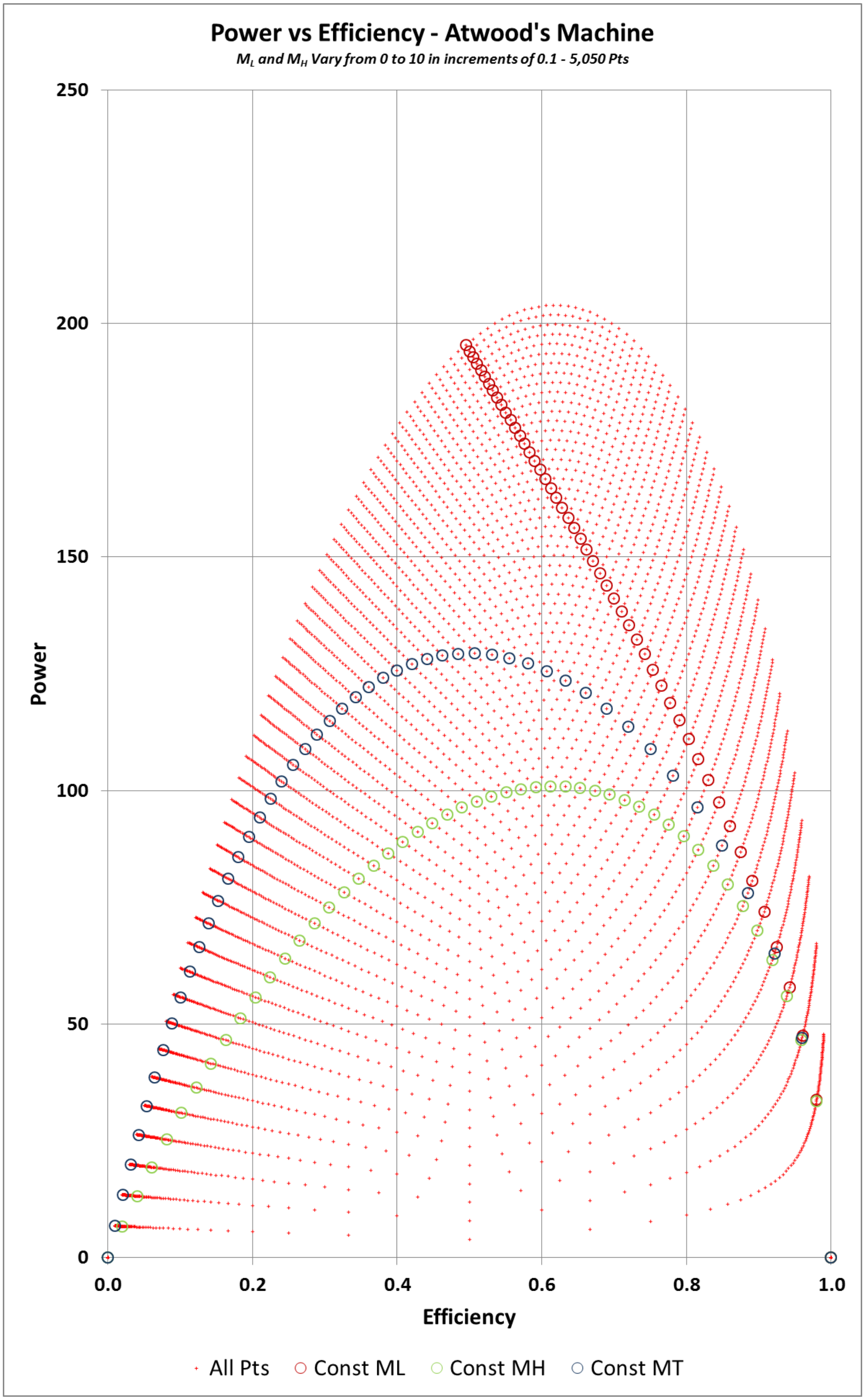




5 Conclusions

I need to understand better where this extra design constraint comes from. Is it real, or is it an arbitrary constraint that is only imposed by us artificially because it produces nice graphs? Since the MPP is evident in nature, there must be natural constraints that force systems to follow such power-efficiency curves. Is it the time regulation constraint that does this?

This family of unit maps seems to be important for understanding the dynamics of such systems. It seems that if parameter A or B are zero, we do not have the conditions for maximum power at an efficiency of intermediate value.


Power vs Efficiency - Atwood's Machine
M_L and M_H Vary from 0 to 10 in increments of 0.1 - 5,050 Pts
250
200
150
100
50
0
Power
0.0
0.2
0.4
0.6
0.8
1.0
Efficiency
All Pts
Const ML
Const MH
Const MT